EXPLICIT ESTIMATES FOR SOME FUNCTIONS OF NUMBER THEORY

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "EXPLICIT ESTIMATES FOR SOME FUNCTIONS OF NUMBER THEORY", submitted by ROBERT A. MacLEOD in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

An integer is called square-free, or quadratfrei, if it is not divisible by the square of any prime. In the first chapter we use known estimates for $M(x) = \sum_{n \leq x} \mu(n)$ to investigate the function $Q(x) = \sum_{n \leq x} |\mu(n)|$, the number of square-free integers less than or equal to x. A typical result is

$$|Q(x) - \frac{6}{\pi^2} x| \leq \sqrt{3} \left(1 - \frac{6}{\pi^2}\right) \sqrt{x}$$

for all x, with equality only at $x = 3$.

In the second chapter, with the help of the results in the first, we obtain a new estimate for $M(x)$, namely

$$|M(x)| < \frac{1}{80} x , \quad \text{for} \quad x \geq 1114 .$$

We apply this estimate to examine the sum $g(x) = \sum_{n \leq x} \frac{\mu(n)}{n}$. In particular we show that whereas $g(x)$ changes sign infinitely often, $g_1(x) = \sum_{14 \leq n \leq x} \frac{\mu(n)}{n}$ is always positive.

In the third chapter, we examine the function $\Phi(x) = \sum_{n \leq x} \varphi(n)$ where $\varphi(n)$ is Euler's function. We show that $\frac{\Phi(x)}{x^2}$, which is

asymptotic to $\frac{3}{\pi^2}$, takes on its minimum (over all positive integers) at $x = 1276$, the second integer x for which $\Phi(x) - \frac{3}{\pi^2} x^2$ is negative.

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TABLE OF CONTENTS

CHAPTER 1

SQUAREFREE INTEGERS

1. Introduction. An integer n is said to be k-free if $d^k | n$ is false for every integer $d \neq \pm 1$ (when $k = 2$, the terms 'square-free' and 'quadratfrei' are normally used). We shall restrict our attention henceforth to positive integers.

Let q_n denote the n^{th} square-free integer, and $Q(x)$ the number of squarefree integers less than or equal to x. Then

$$Q(x) = \sum_{n \leq x} |\mu(n)| \quad ,$$

where $\mu(n)$ denotes the Möbius function. We shall give a brief survey of known results on q_n and $Q(x)$, and then proceed to examine the difference $Q(x) - \frac{6}{\pi^2} x$.

2. Survey of results. From the simple result that

$$Q(x) = \frac{6}{\pi^2} x + O(x^{1/2}) \tag{1.1}$$

(we shall prove stronger results later) it follows that

$$q_{n+1} - q_n = O(n^{1/2}) \quad . \tag{1.2}$$

In 1941, E. Fogels [8] improved this to

$$q_{n+1} - q_n = O(n^{2/5 + \epsilon}) \qquad \text{for every} \quad \epsilon > 0 . \tag{1.3}$$

K. F. Roth [22], using an argument which he attributed to Estermann, showed that

$$q_{n+1} - q_n = O(n^{1/3}) \tag{1.4}$$

and upon strengthening the argument was able to obtain

$$q_{n+1} - q_n = O(n^{3/13} (\log n)^{4/13}) . \tag{1.5}$$

H.-E. Richert [20] improved this to

$$q_{n+1} - q_n = O(n^{2/9} \log n) . \tag{1.6}$$

Using slightly different arguments R. A. Rankin [19] obtained

$$q_{n+1} - q_n = O(n^{\gamma+\epsilon}) \qquad \text{where} \quad \gamma = 0.22198 \ldots . \tag{1.7}$$

H. Halberstam and Roth [10] have generalized the argument giving $q_{n+1} - q_n = O(n^{1/3})$ to obtain the following: if $q_k(n)$ denotes the n^{th} k-free number, then

$$q_k(n+1) - q_k(n) = O(n^{1/(k+1)}) . \tag{1.8}$$

In the other direction, P. Erdos [4] has shown that, for infinitely many i ,

$$q_{i+1} - q_i > (1 + o(1)) \frac{\pi^2}{6} \frac{\log q_i}{\log \log q_i} \quad . \tag{1.9}$$

He states that it appears to be very difficult to improve on the constant $\frac{\pi^2}{6}$, and suggests that it may indeed be that, for $i > i_o$,

$$q_{i+1} - q_i < (1+\epsilon) \frac{\pi^2}{6} \frac{\log q_i}{\log \log q_i} \quad \text{for every} \quad \epsilon > 0 \ . \tag{1.10}$$

He also conjectures that, for every α,

$$\sum_{q_{k+1} < x} (q_{k+1} - q_k)^{\alpha} = c_{\alpha} x + o(x) \quad . \tag{1.11}$$

This, if true, would imply

$$q_{n+1} - q_n = O(n^{\epsilon}) \ , \quad \text{for every} \quad \epsilon > 0 \quad , \tag{1.12}$$

but he can prove it only for $\alpha < A$, where A is a constant between 2 and 3 .

R. Bellman and H. N. Shapiro [1] consider a slightly different problem, and obtain the following results:

1. If $\varphi(x)$ is any function such that $\lim_{x \to \infty} \varphi(x) = \infty$, then for almost all n the interval $(n, n+\varphi(n))$ contains a squarefree integer.

2. If $\varphi(x)$ is any strictly monotone function such that $\lim_{x \to \infty} \varphi(x) = \infty$, then $Q(n, n+\varphi(n))$ has normal order $\frac{6}{\pi^2} \varphi(n)$; that is, for every $\epsilon > 0$ and $n > n(\epsilon)$,

$$(1-\epsilon)\frac{6}{\pi^2}\varphi(n) < Q(n,n+\varphi(n)) < (1+\epsilon)\frac{6}{\pi^2}\varphi(n) \tag{1.13}$$

where, of course, $Q(n,n+\varphi(n))$ denotes the number of squarefree integers between n and $n+\varphi(n)$.

In a different direction, E. Cohen and R. L. Robinson [3] have examined the distribution of k-free integers in residue classes. They have shown that the k-free integers are equi-distributed (mod h) if and only if every prime factor of h divides h at least to the k^{th} power. (A set of integers is said to be equi-distributed (mod h) if the density is the same in each residue class (mod h) where the integers occur at all.)

T. Estermann [6] found that the number of representations of n as the sum of two square-free integers is

$$cn\rho(n) + O(n^{2/3+\epsilon}) \tag{1.14}$$

where $c = \prod_p (1-\frac{2}{p^2})$ and $\rho(n) = \prod_{p^2|n} (1+\frac{1}{p^2-2})$. Cohen [2] was able to improve the error term to $O(n^{2/3}\log^2 n)$.

Considerable work has been done on pattern problems for square-free integers. W. Sierpinski [26] showed in 1959 that, for infinitely many k , the integers $4k+1$, $4k+2$, and $4k+3$ are all square-free. I was able to generalize this to show that any pattern of square-free and non-square-free integers which occurs at all in the sequences of integers occurs infinitely often. However, considerably sharper results

had been discovered by S. S. Pillai [17] in 1936 and L. Mirsky [14], [15] in 1948 and 1949, but apparently overlooked by Sierpinski. They are the following.

Let $d_1, d_2, \ldots, d_{r-1}$ be a fixed set of positive integers, and for any prime p let $g(p)$ be the number of different residue classes mod p^2 among $0, d_1, d_2, \ldots, d_{r-1}$. Let $N(x)$ be the number of positive integers t such that $t, t+d_1, \ldots, t+d_{r-1}$ are all squarefree and $\leq x$. Pillai showed that

$$N(x) = Ax + O(\frac{x}{\log x}) \tag{1.15}$$

where $A = \prod_p (1 - \frac{g(p)}{p^2})$. In particular, for $r = 2$, $d_1 = 1$, $d_2 = 2$ (i.e. for Sierpinski's problem) $A = \prod_p (1 - \frac{3}{p^2}) \approx 0.12$ so that for about half of all k's the integers $4k+1$, $4k+2$, and $4k+3$ are all square-free.

Mirsky generalized Pillai's results as follows: let $a_1, \ldots, a_\ell;\ b_1, \ldots, b_m$ be any distinct positive integers; let $H(x) = H_r(x; a_1, \ldots, a_\ell,\ b_1, \ldots, b_m)$ be the number of systems of positive integers $n+a_1, \ldots, n+a_\ell;\ n+b_1, \ldots, n+b_m$ not exceeding x and such that the first ℓ are r-free while the remaining m are not. Then

$$H(x) = hx + O(x^{2/(r+1) + \epsilon}) , \tag{1.16}$$

$$\text{where } h = \begin{cases} 0\ , & \text{if } D(p^r;\, a_1,\dots,a_\ell) = p^r \text{ for some } p \\ h_r(a_1,\dots,a_\ell;\, b_1,\dots,b_m) = \sum\limits_{k=0}^{m} (-1)^k \sum\limits_{1 \le v_1 < \dots < v_k \le m} \prod\limits_p \\ \quad \left\{1 - \dfrac{D(p^r\,;\, a_1,\dots,a_\ell;\, b_{v_1},\dots,b_{v_k})}{p^r}\right\} & \text{otherwise,} \end{cases}$$

and $D(\sigma;\, n_1,\dots,n_s)$ denotes the number of different residue classes mod σ represented by $n_1, n_2, \dots, n_s$. As a special case, consider blocks. By a block of s integers with respect to a class C , we mean a sequence of s consecutive integers, say $n, n+1, \dots, n+s-1$, in C , while $n-1$ and $n+s$ are not in C . Let $Q_{r,s}(x)$ denote the number of blocks of s r-free integers $\le x$ and $V_{r,s}(x)$ the number of blocks of s r-integers (i.e. non-r-free integers) $\le x$. Then

(i) for $r \ge 2,\ s \ge 2^r,\ Q_{r,s}(x) = 0$;

(ii) for $r \ge 2,\ 1 \le s \le 2^r-1,\ \ Q_{r,s}(x) = q_{r,s}x + O(x^{2/(r+1)+\epsilon})$

where

$$q_{r,s} = \begin{cases} \prod\limits_p (1-\frac{s}{p^r}) - 2\prod\limits_p (1-\frac{s+1}{p^r}) + \prod\limits_p (1-\frac{s+2}{p^r}), & 1 \le s \le 2^r-2\ , \\ \prod\limits_p (1-\frac{2^r-1}{p^r})\ , & s = 2^r-1\ , \end{cases}$$

$(q_{r,s} > 0)$;

(iii) for $r \ge 2,\ s \ge 1,\ V_{r,s}(x) = v_{r,s}x + O(x^{2/(r+1)+\epsilon})$ where

$$v_{r,s} = \sum_{k=o}^{s} (-1)^k g(k) \prod_{p > (s+1)^{1/r}} (1-\frac{k+2}{p^r})$$

and

$$g(k) = g_{r,s}(k) = \sum_{1\leq v_1<\ldots<v_s\leq s} \prod_{p\leq(s+1)^{1/r}} \left\{1 - \frac{D(p^r;0,v_1,\ldots,v_r s+1)}{p^k}\right\}$$

$(v_{r,s} > 0)$.

In particular, for fixed t the number of $q_i < x$ satisfying $q_{i+1}-q_i = t$ is known.

We remark also that K. Rogers [21] has shown that

$$\frac{Q(x)}{x} \geq \frac{53}{88}, \text{ with equality only at } x = 176 \quad , \qquad (1.17)$$

that is, the Schnirelmann density of the square-free integers is less than their density, $\frac{6}{\pi^2}$.

3. <u>The difference $Q(x) - \frac{6}{\pi^2} x$</u> . Define $\mu_r(n)$ to be 0 or 1 according as n is or is not r-free, for $r = 2, 3, \ldots$. Then in particular $\mu_2(n) = |\mu(n)| = \mu^2(n)$. Define $Q_r(x)$ by

$$Q_r(x) = \sum_{n \leq x} \mu_r(n) \quad .$$

In particular, $Q_2(x) = Q(x)$. Then, letting $[x]$ denote the greatest

integer $\leq x$, we have

$$Q_r(x) = [x] - \left[\frac{x}{2^r}\right] - \left[\frac{x}{3^r}\right] - \left[\frac{x}{5^r}\right] - \left[\frac{x}{7^r}\right] - \ldots + \left[\frac{x}{(2.3)^r}\right] + \left[\frac{x}{(2.5)^r}\right] +$$

$$\ldots - \left[\frac{x}{(2.3.5)^r}\right] - \ldots$$

$$= \sum_{d^r \leq x} \mu(d) \left[\frac{x}{d^r}\right] . \tag{1.18}$$

Alternately, since $\displaystyle\sum_{d^r | n} \mu(d) = \begin{cases} 1, & \text{if } n \text{ is } r\text{-free}, \\ 0 & \text{otherwise}, \end{cases}$

(for if $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k} n_1$, where $\alpha_i \geq r$ and n_1 is r-free, then the sum is $1 - \binom{k}{1} + \binom{k}{2} - \ldots = (1-1)^k = 0$), we have

$$\sum_{d^r | n} \mu_r(d) = \mu_r(n) .$$

Thus

$$Q_r(x) = \sum_{n \leq x} \sum_{d^r | n} \mu(d) = \sum_{kd^r \leq x} \mu(d) = \sum_{d^r \leq x} \mu(d) \Bigg(\sum_{\substack{n \leq x \\ d^r | n}} 1 \Bigg)$$

$$= \sum_{d^r \leq x} \mu(d) \left[\frac{x}{d^r}\right] . \tag{1.19}$$

Therefore,

$$|Q_r(x) - \frac{1}{\zeta(r)} x| \le |\sum_{d^r \le x} \mu(d) \left\{\frac{x}{d^r}\right\}| + x|\sum_{d^r > x} \frac{\mu(d)}{d^r}|, \qquad (1.20)$$

where $\{x\} = x - [x]$ is the fractional part of x. Hence, defining $R_r(x)$ by $R_r(x) = Q_r(x) - \frac{1}{\zeta(r)} x$, we have

$$|R_r(x)| \le x^{1/r} + \frac{1}{r-1} x^{1/r} + 1 = \frac{r}{r-1} x^{1/r} + 1 , \qquad (1.21)$$

since

$$\sum_{d^r > x} \frac{1}{d^r} < \frac{1}{x} + \int_{x^{1/r}}^{\infty} \frac{1}{u^r} = \frac{1}{x} + \frac{1}{r-1} \frac{x^{1/r}}{x} .$$

For $r = 2$, the best improvement in the constant 2 which appears in (1.21) was $\frac{12}{\pi^2}$ by Rogers [21]. Our main object in this chapter will be to obtain further improvements on this.

Let $M(x) = \sum_{n \le x} \mu(n)$. R. D. von Sterneck [22] showed that

$$|M(x)| < \frac{1}{9} x + 8 \qquad (1.22)$$

and R. Hackel [9] improved this to

$$|M(x)| < \frac{1}{26} x + 155 , \quad \text{for all } x . \qquad (1.23)$$

G. Neubauer [16] has shown that

$$|M(x)| < \frac{1}{2}\sqrt{x} \ , \qquad \text{for} \quad 200 < x \leq 10^8 \ . \tag{1.24}$$

Combining (1.23) and (1.24) we obtain

$$|M(x) + 2| < \frac{1}{25}\, x \ , \qquad \text{for} \quad x > 200 \ , \tag{1.25}$$

and one readily checks that (1.25) holds for $x \geq 100$. Thus, for $x \geq 100^r$ and $r > 1$,

$$\Big|\sum_{d^r > x} \frac{\mu(d)}{d^r}\Big| = \Big|\sum_{d^r > x} (M(d) + 2)\left(\frac{1}{d^r} - \frac{1}{(d+1)^r}\right) - \frac{M(x^{1/r}) + 2}{([x^{1/r}]+1)^r}\Big|$$

$$\leq \frac{1}{25} \sum_{d^r > x} d\left(\frac{1}{d^r} - \frac{1}{(d+1)^r}\right) + \frac{1}{25} \ \frac{x^{1/r}}{x}$$

$$\leq \frac{1}{25}\left(\frac{x^{1/r}}{x} + \frac{1}{r-1} \ \frac{x^{1/r}}{x} + \frac{x^{1/r}}{x}\right) = \frac{1}{25}\ (2 + \frac{1}{r-1})\ \frac{x^{1/r}}{x^r} \ .$$

Therefore,

$$x\Big|\sum_{d^r > x} \frac{\mu(d)}{d^r}\Big| \leq \frac{1}{25}\ (2 + \frac{1}{r-1})\ x^{1/r} \ , \qquad \text{for} \quad x \geq 100^r \ . \tag{1.26}$$

Let

$$A(x) = \sum_{\substack{d \leq x \\ \mu(d)=1}} 1 \ , \quad B(x) = \sum_{\substack{d \leq x \\ \mu(d)=-1}} 1 \ , \quad \text{and} \quad C(x) = \max\,(A(x),B(x)) \ .$$

Then clearly

$$\left| \sum_{d \leq x} \mu(d) \{g(x,d)\} \right| \leq C(x) \qquad \text{for any } g \quad .$$

Now

$$C(x) = \frac{1}{2}(A(x) + B(x)) + \frac{1}{2}|A(x) - B(x)| = \frac{1}{2}Q(x) + \frac{1}{2}M(x) \; .$$

Thus

$$\left| \sum_{d \leq x} \mu(d) \{g(x,d)\} \right| \leq \frac{1}{2}Q(x) + \frac{1}{2}M(x) \quad , \tag{1.27}$$

whence

$$\left| \sum_{d^r \leq x} \mu(d) \left\{ \frac{x}{d^r} \right\} \right| \leq \frac{13}{25} x^{1/r} + 1 \quad , \tag{1.28}$$

and

$$|R_r(x)| \leq \frac{16}{25} x^{1/r} + 1 \, , \qquad x \geq 100^r \quad , \tag{1.29}$$

by (1.28) and (1.26). In particular,

$$\begin{aligned} Q(x) - \frac{6}{\pi^2} x &\leq \frac{16}{25} x^{1/2} + 1 \\ &\leq 0.65 \, x^{1/2} \, , \qquad \text{for} \quad x \geq 100^2 \quad . \end{aligned}$$

Using this in (1.27) we obtain

$$C(x) \leq \left(\frac{3}{\pi^2} + \frac{1}{50} \right) x + 0.325 \, x^{1/2} \, , \qquad \text{for} \quad x \geq 100^2 \quad .$$

Hence,

$$\left| \sum_{d^r \leq x} \mu(d) \left\{ \frac{x}{d^r} \right\} \right| < \frac{1}{3} x^{1/r} + 0.325 x^{1/2r} , \quad \text{for} \quad x \geq 100^r \quad (1.30)$$

and

$$|R_r(x)| \leq \frac{28}{75} x^{1/r} + 0.325 x^{1/2r} , \quad \text{for} \quad x \geq 100^r . \quad (1.31)$$

For $r = 2$, using (1.31) and examining the early cases, we find that

$$|R_2(x)| \leq \sqrt{3} \left(1 - \frac{6}{\pi^2}\right) x^{1/2} \quad \text{for all} \quad x , \quad (1.32)$$

with equality only at $x = 3$ $(\sqrt{3}(1-\frac{6}{\pi^2}) = 0.679 \ldots)$, and that

$$|R_2(x)| < \frac{1}{2} x^{1/2} \quad \text{for} \quad x \geq 8 . \quad (1.33)$$

The first place where $R_2(x)$ becomes negative is at $x = 28$. A. M. Vaidya [29] apparently has shown (the details have not yet been published) that $R_2(x)$ changes sign infinitely often, and that in fact for every $\epsilon > 0$ and infinitely many n, $R_2(n) > n^{1/4 - \epsilon}$, and for infinitely many n, $R_2(n) < -n^{1/4 - \epsilon}$; from this it follows that, for infinitely many n, $Q(n) = [\frac{6}{\pi^2} n+1]$.

We have seen that

$$R_2(x) = O(x^{1/2}) . \quad (1.34)$$

It is well known that $M(x) = O\left(\frac{x}{\log^{\alpha} x}\right)$ for arbitrary α (see e.g. E. Landau [11], p. 57). One can readily show from this that

$$R_2(x) = O\left(\frac{x^{1/2}}{\log^\alpha x}\right) \quad \text{for arbitrary } \alpha . \tag{1.35}$$

In the opposite direction, C. J. A. Evelyn and E. H. Linfoot [7] have shown that

$$R_2(x) \neq o(x^{1/4}) . \tag{1.36}$$

CHAPTER 2

On the Sum $M(x) = \sum_{n \leq x} \mu(n)$

1. Introduction. Our object here will be to obtain the result

$$|M(x)| < \frac{x}{80} \quad \text{for} \quad x \geq 1114 \quad , \tag{2.1}$$

an improvement of the results already cited of von Sterneck [27] and Hackel [9]. We first observe that, from Neubauer's [16] result that

$$|M(x)| < \frac{1}{2}\sqrt{x} \quad \text{for} \quad 200 < x \leq 10^8 \quad , \tag{2.2}$$

we can prove (2.1) for $1114 \leq x \leq 10^8$. For since $\frac{1}{2}\sqrt{x} < \frac{x}{80}$ for $x > 1600$, (2.1) follows for $1600 < x \leq 10^8$, and one obtains by simple checking that (2.1) also holds for $1114 \leq x \leq 1600$; thus, (2.1) remains to be verified for $x > 10^8$.

2. The method. We outline the method to be used, which is a refinement of that of von Sterneck. Consider the function

$$f(x) = [x] - [\tfrac{x}{2}] - [\tfrac{x}{3}] - [\tfrac{x}{5}] + [\tfrac{x}{30}] \ .$$

Since

$$\sum_{d \leq x} \mu(d) \left[\frac{x}{d}\right] = 1 \quad ,$$

we have

$$\sum_{d \leq x} \mu(d) \left[\frac{x}{md}\right] = \sum_{d \leq \frac{x}{m}} \mu(d) \left[\frac{x}{md}\right] + \sum_{\frac{x}{m} < d \leq x} \mu(d) \left[\frac{x}{md}\right]$$

$$= 1 + 0 = 1 \ , \qquad \text{for} \quad x \geq m \quad ,$$

and thus

$$\sum_{d \leq x} \mu(d) \ f\left(\frac{x}{d}\right) = -1 \quad , \quad \text{for} \quad x \geq 30 \quad .$$

Since $f(x) = 1$ for $1 \leq x < 6$ and 0 or 1 for $x \geq 6$, we have $f(\frac{x}{d}) = 1$ for $d > \frac{x}{6}$, so that

$$\left| \sum_{d \leq x} \mu(d)\left(1 - f\left(\frac{x}{d}\right)\right) \right| \leq \sum_{d \leq \frac{x}{6}} |\mu(d)| = Q\left(\frac{x}{6}\right) \quad .$$

Thus,

$$|M(x) + 1| \leq Q\left(\frac{x}{6}\right) \quad , \quad \text{for} \quad x \geq 30 \ . \tag{2.3}$$

We have from Chapter 1 that

$$\left|Q(x) - \frac{6}{\pi^2} x\right| < \frac{1}{2}\sqrt{x} \quad , \quad \text{for} \quad x \geq 8 \quad . \tag{2.4}$$

It follows that

$$0.600x < Q(x) < 0.615x \quad \text{for} \quad x \geq 5000 \quad , \tag{2.5}$$

and one readily checks that (2.5) holds for $x \geq 475$. Similarly,

$$Q(x) < 0.635x \quad \text{for} \quad x \geq 75 \ . \tag{2.6}$$

Using (2.5) in (2.3) we obtain

$$|M(x) + 1| \leq 0.103x \quad \text{for} \quad x \geq 2950 \tag{2.7}$$

and, by (2.2), for $x > 200$. If we further observe that $f(x) = 1$ for $7 \leq x < 10$, we have

$$\begin{aligned} |M(x) + 1| &\leq Q(\tfrac{x}{6}) - Q(\tfrac{x}{7}) + Q(\tfrac{x}{10}) \\ &< 0.079x \ , \quad \text{for} \quad x > 200 \ . \end{aligned} \tag{2.8}$$

Using the function

$$f_1(x) = [x] - [\tfrac{x}{2}] - [\tfrac{x}{3}] - [\tfrac{x}{5}] + [\tfrac{x}{15}] - [\tfrac{x}{30}]$$

and similar refinements to that used in deriving (2.8), one can obtain

$$|M(x) + 2| < 0.04x \ , \quad \text{for} \quad x > 200 \ , \tag{2.9}$$

which is the same as (1.25). It seems difficult to get a fairly simple function like $f_1(x)$ which will substantially improve (2.9) .

If we examine the characteristics of a 'good' function f, we see that what we would like is a function which takes the value 1 for $1 \leq x \leq n$ for fairly large n , and then does not differ too widely from 1 thereafter. We shall employ the techniques of E. Waage ([30] and [31]) to obtain such a function.

3. <u>Main result</u>. In line with Waage, we define

$$u_k(x) = [\tfrac{x}{k}] - [\tfrac{x}{k+1}] - [\tfrac{x}{k(k+1)}]$$

and use the symbol

$$(n_1, n_2, \ldots, n_m; \ \ell_1, \ell_2, \ldots, \ell_t)$$

to stand for the function

$$\left[\frac{x}{n_1}\right] + \left[\frac{x}{n_2}\right] + \ldots + \left[\frac{x}{n_m}\right] - \left[\frac{x}{\ell_1}\right] - \left[\frac{x}{\ell_2}\right] - \ldots - \left[\frac{x}{\ell_t}\right] \quad .$$

Let

$$U_2(x) = u_1(x) = (1;\ 2,\ 2) \quad ,$$

$$U_5(x) = u_1(x) + u_2(x) = (1;\ 2,\ 3,\ 6) \quad ,$$

$$U_6(x) = U_5(x) - u_5(x) = (1,\ 30;\ 2,\ 3,\ 5) \qquad \text{(this is our } f(x)) \quad ,$$

$$U_{10}(x) = U_6(x) + u_6(x) = (1,\ 6,\ 30;\ 2,\ 3,\ 5,\ 7,\ 42) \quad .$$

Define $U_s(x)$, $U_f(x)$, $u(x)$, and $u'(x)$, respectively, as follows:

$$U_s(x) = U_{10}(x) - u_5(\tfrac{x}{6}) - u_6(\tfrac{x}{6}) + u_2(\tfrac{x}{35}) - u_1(\tfrac{x}{105}) - u_6(\tfrac{x}{30}) - u_5(\tfrac{x}{42})$$

$$= (1,\ 6,\ 70;\ 2,\ 3,\ 5,\ 7,\ 210) \quad ,$$

$$U_f(x) = U_s(x) + u_{10}(x) - u_2(\tfrac{x}{35}) - u_{21}(\tfrac{x}{5})$$

$$= (1,\ 6,\ 10,\ 2310;\ 2,\ 3,\ 5,\ 7,\ 11) \quad ,$$

$$u(x) = U_s(x) + u_1(\tfrac{x}{10}) + u_2(\tfrac{x}{10}) + u_1(\tfrac{x}{30}) + u_1(\tfrac{x}{14}) + 2u_1(\tfrac{x}{28})$$

$$+ 4u_4(\tfrac{x}{14}) - u_1(\tfrac{x}{70}) - 2u_1(\tfrac{x}{140}) + u_1(\tfrac{x}{21}) + u_1(\tfrac{x}{35}) + u_2(\tfrac{x}{35})$$

$$= (1,6,10,14,21,35;2,3,5,7,30,30,30,42,42,70,70,70,70,70,105,210,210),$$

$$u'(x) = u_2(\tfrac{x}{5}) + u_6(\tfrac{x}{5}) - u_{14}(x) = (10;\ 14,35) \ .$$

Let R_1, R_2, R_3, and R_4 be respectively the sets

$\{11,13,17,19,23,29,31,37,41,43,47\}$, $\{18,70,90,90,118,134,142,146,162,177,183,213\}$,

$\{113,131,139,154,170,173,191\}$, $\{18,18,54,90,90,105,107,108,109,700,700,700,700\}$,

Then the function $e(x)$ which we shall use is defined by the formula

$$e(x) = u(x) - \sum_{r \in R_1} U_f(\tfrac{x}{r}) + u'(\tfrac{x}{7}) + 2u'(\tfrac{x}{14}) + u'(\tfrac{x}{17}) + U_5(\tfrac{x}{42}) - U_5(\tfrac{x}{79})$$

$$- U_5(\tfrac{x}{101}) - U_5(\tfrac{x}{103}) - U_5(\tfrac{x}{137}) - U_5(\tfrac{x}{163}) - U_5(\tfrac{x}{167}) + 2U_6(\tfrac{x}{30}) + \sum_{r \in R_2} u_1(\tfrac{x}{r})$$

$$- \sum_{r \in R_3} u_1(\tfrac{x}{r}) + \sum_{r \in R_4} u_2(\tfrac{x}{r}) - 2u_2(\tfrac{x}{1400}) + u_3(\tfrac{x}{18}) - u_3(\tfrac{x}{80}) + u_3(\tfrac{x}{108}) + 3u_4(\tfrac{x}{108})$$

$$+ u_5(\tfrac{x}{3}) - 3u_8(\tfrac{x}{200}) + 6u_9(\tfrac{x}{60}) - 3u_9(\tfrac{x}{115}) + u_{10}(\tfrac{x}{21}) - 3u_{10}(\tfrac{x}{46}) + 3u_{10}(\tfrac{x}{92})$$

$$- 3u_{10}(\tfrac{x}{103}) - 2u_{14}(\tfrac{x}{20}) + u_{18}(\tfrac{x}{18}) - 2u_{19}(\tfrac{x}{23}) + u_{21}(\tfrac{x}{14}) - 2u_{23}(\tfrac{x}{10}) - u_{26}(\tfrac{x}{12})$$

$$+ u_{27}(\tfrac{x}{11}) + 3u_{33}(\tfrac{x}{10}) - u_{34}(\tfrac{x}{2}) + u_{44}(\tfrac{x}{5}) - 2u_{48}(\tfrac{x}{5}) + u_{49}(\tfrac{x}{8}) + u_{50}(\tfrac{x}{5}) -$$

$$- u_{50}(\tfrac{x}{10}) - u_{52}(\tfrac{x}{10}) - u_{53}(x) - 2u_{58}(\tfrac{x}{100}) - u_{59}(x) + u_{60}(x) + 3u_{60}(\tfrac{x}{10})$$

$$- u_{67}(x) + 3u_{67}(\tfrac{x}{100}) + u_{68}(\tfrac{x}{5}) - u_{68}(\tfrac{x}{10}) + u_{70}(x) + u_{72}(x) - u_{81}(\tfrac{x}{20})$$

$$- u_{83}(x) - u_{89}(x) + u_{90}(\tfrac{x}{2}) + u_{94}(\tfrac{x}{10}) - u_{96}(\tfrac{x}{3}) - u_{97}(x) + u_{106}(x)$$

$$+ u_{108}(x) + 2u_{109}(\tfrac{x}{3}) - u_{114}(\tfrac{x}{2}) - u_{121}(x) + u_{126}(x) - u_{139}(\tfrac{x}{2}) - u_{143}(\tfrac{x}{2})$$

$$- u_{144}(\tfrac{x}{20}) - u_{149}(x) + u_{150}(x) - u_{157}(x) - u_{158}(x) - u_{165}(x) + u_{178}(x)$$

$$+ u_{180}(x) - u_{193}(x) - u_{195}(x) + u_{196}(x) - u_{199}(x) - u_{200}(x) + u_{210}(x)$$

$$- u_{263}(x) - u_{264}(x) + u_{264}(\tfrac{x}{10}) - u_{266}(x) + u_{270}(x) + 3u_{270}(\tfrac{x}{20}) - u_{283}(x)$$

$$- u_{320}(x)$$

$$= \sum_{n=1}^{218} \mu(n)[\tfrac{x}{n}] + \sum_{p \in P} [\tfrac{x}{p}] - \sum_{q \in Q} [\tfrac{x}{q}] \quad ,$$

where

P = {220,226,226,235,237,245,250,253,259,262,262,265,267,274,278,
287,291,294,297,300,300,301,303,309,319,326,327,329,330,334,
340,341,346,346,382,382,392,407,411,451,473,474,489,501,506,
506,506,510,517,530,540,540,540,540,606,618,690,690,690,720,
720,720,800,800,822,920,920,920,940,960,978,1002,1133,1133,1133,
1150,1150,1150,1200,1200,1400,1400,1400,1400,1640,1800,1800,1800,
2380,2640,2862,2900,3540,4556,5060,5060,5060,5520,5520,5900,5900,

6700,6700,6700,6972,8010,8400,8400,8424,8740,8740,9506,10350,10350,
10350,11330,11330,11330,11760,11760,13200,13200,14400,14400,14400,
14762,15180,15180,16002,22350,24806,25122,25500,26220,27390,27560,
27936,37442,38220,38920,39800,40200,41184,46920,66420,69432,69960,
71022,80372,102720,342200,342200,417600}

(2.10)

and

Q = {222,225,228,230,230,231,236,236,238,240,246,252,255,258,263,266,
268,268,270,271,280,282,283,284,286,290,292,292,310,312,315,324,
342,345,345,354,354,366,366,370,400,410,426,426,430,432,437,437,
460,470,490,490,500,520,595,600,600,660,680,820,820,900,950,1012,
1012,1012,1030,1030,1030,1035,1035,1035,1100,1100,1296,1600,1600,
1600,1620,2100,2100,2100,2100,2310,2650,2800,2800,2880,3600,3600,
3600,3660,4970,5256,5400,5400,5400,5420,5420,5420,5800,5800,6156,
6468,6800,6800,6800,8316,9900,10120,10120,10120,11342,11220,11220,
11220,11772,12750,19600,22650,23460,23460,25410,30030,31862,32580,
35970,35970,36600,36600,36600,38612,39270,43890,44310,53130,66990,
71610,73170,85470,89300,94710,99330,108570,455600,455600,455600,
699600,1463400,1463400,1463400}.

This rather complicated function was obtained by successively evaluating simpler functions by computer to see where they began to differ too much from 1, and adding in compensating simple functions to reduce the rate of growth.

Since there are 222 positive terms and 226 negative terms in g,

$$|e(x)| \leq 226 \qquad \text{for all } x , \qquad (2.11)$$

for when we remove the square brackets in e the function is identically zero by construction. Upon examining $e(x)$, we find that

$$e(x) = 1 , \quad \text{for} \quad 1 \leq x < 219 \qquad (2.12)$$

and

$$|e(x)-1| \leq k \quad \text{for} \quad x < n , \qquad (2.13)$$

where k and n are as given in the following table:

k	1	2	3	4	5	6	7	8	9	10	11
n	345	568	584	804	1237	1359	1391	1393	1416	1416	1417

k	12	13	14	15	16	17	18	19	20
n	5010	5011	5881	5882	16097	16100	16100	16103	26740

k	21	22	23	24	25	26	27	28	29
n	26750	26752	26754	26759	31397	46110	46110	46112	63611

k	30	31	32	33	34	35	36	37	38
n	67158	67159	67189	69258	69259	69263	82800	82800	82813

k	39	40	41	42	43	44	45
n	85869	87542	87547	97006	97007	106591	up to 125000 .

Let N be the set of n's in the above table. It follows that

$$|M(x)+4| \leq Q(\frac{x}{219}) + \sum_{n \in N} Q(\frac{x}{n}) + (226 - 45)\, Q(\frac{x}{12500}) \quad . \tag{2.14}$$

Using (2.4) and (2.5) in (2.14) we obtain

$$|M(x)+4| \leq 0.01247\, x \,, \qquad \text{for} \quad x > 10^8$$

or

$$|M(x)| < \frac{1}{80}\, x \,, \qquad \text{for} \quad x > 10^8 \quad .$$

This completes the proof of (2.1).

It is likely that with a better function $e(x)$ one could prove

$$|M(x)| < .01\, x \,, \qquad \text{for} \quad x \geq 1137 \quad .$$

This is certainly true for $1137 \leq x \leq 10^8$.

4. Application. S. Selberg [25] has shown that, if $g(x)$ is defined by

$$g(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \quad ,$$

then $g(x)$ changes sign infinitely often. We show here that, on the other hand, $g_1(x)$, defined by

$$g_1(x) = \sum_{14 \le n \le x} \frac{\mu(n)}{n} = g(x) - \sum_{n \le 13} \frac{\mu(n)}{n} ,$$

is always positive, or, what is the same thing, that $g(x)$ has its minimum at $x = 13$.

We note that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \tag{2.15}$$

(see e.g. Landau [11], page 159) and observe that

$$\sum_{n \le 13} \frac{\mu(n)}{n} = -0.0773559 \ldots \quad . \tag{2.16}$$

To show that

$$\left| \sum_{n \le x} \frac{\mu(n)}{n} \right| < 0.07 \qquad \text{for} \quad 200 < n \le 10^8$$

we can use $|M(x)| < \frac{1}{2}\sqrt{x}$. Then, for $900 \le x < 10^8$ we have

$$\sum_{d \le x} \frac{\mu(d)}{d} = \sum_{d \le 900} \frac{\mu(d)}{d} + \sum_{900 < d \le x-1} \frac{M(d)}{d(d+1)} - \frac{M(900)}{901} + \frac{M(x)}{x} \quad .$$

Since

$$\sum_{d \leq 900} \frac{\mu(d)}{d} = 0.00328 \ldots \quad \text{and} \quad M(900) = 1$$

we obtain

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| = 0.00217 \ldots + \frac{1}{2} \sum_{900 < d \leq x-1} \frac{1}{d^{3/2}} + \frac{1}{2} \frac{1}{x^{1/2}}$$

$$< 0.036 , \quad \text{for} \quad 900 \leq x \leq 10^8 . \tag{2.17}$$

One readily checks $g(x)$ for $1 \leq x \leq 900$, and we have that, for $1 \leq x \leq 10^8$, g assumes its minimum only at $x = 13$. So it remains only to check for $x > 10^8$.

Since

$$\sum_{d \leq x} \mu(d) \left[\frac{x}{d}\right] = 1 ,$$

we have

$$\sum_{d \leq x} \frac{\mu(d)}{d} = \frac{1}{x} + \frac{1}{x} \sum_{d \leq x} \mu(d) \left\{\frac{x}{d}\right\} . \tag{2.18}$$

Now,

$$\sum_{d\le x}\mu(d)\left\{\frac{x}{d}\right\} = \sum_{\frac{x}{2}<d\le x}\mu(d)\left(\frac{x}{d}-1\right) + \sum_{\frac{x}{3}<d\le\frac{x}{2}}\mu(d)\left(\frac{x}{d}-2\right) + \dots + \sum_{\frac{x}{k}<d\le\frac{x}{k-1}}\mu(d)\left(\frac{x}{d}-(k-1)\right)$$

$$+ \sum_{d\le\frac{x}{k}}\mu(d)\left\{\frac{x}{d}\right\}$$

$$= x\sum_{\frac{x}{k}<d\le x}\frac{\mu(d)}{d} - \sum_{1\le t\le k-1}M\left(\frac{x}{t}\right) + (k-1)M\left(\frac{x}{k}\right) + \sum_{d\le\frac{x}{k}}\mu(d)\left\{\frac{x}{d}\right\} .$$

Therefore, using (2.18), it follows that

$$\sum_{d\le\frac{x}{k}}\frac{\mu(d)}{d} = \frac{1}{x} - \frac{1}{x}\sum_{1\le t\le k-1}M\left(\frac{x}{t}\right) + \frac{k-1}{x}M\left(\frac{x}{k}\right) + \frac{1}{x}\sum_{d\le\frac{x}{k}}\mu(d)\left\{\frac{x}{d}\right\} , \quad x\ge k .$$

Hence,

$$\sum_{d\le x}\frac{\mu(d)}{d} = \frac{1}{kx} - \frac{1}{kx}\sum_{1\le t\le k-1}M\left(k\,\frac{x}{t}\right) + \frac{k-1}{kx}M(x) + \frac{1}{kx}\sum_{d\le x}\mu(d)\left\{\frac{kx}{d}\right\}, \quad \text{for} \quad x\ge 1 . \tag{2.19}$$

Using (2.1) in (2.19), we obtain

$$\left|\sum_{d\le x}\frac{\mu(d)}{d}\right| \le \frac{1}{kx} + \frac{1}{80}\sum_{1\le t\le k-1}\frac{1}{t} + \frac{1}{80}\,\frac{k-1}{k} + \frac{1}{kx}\left|\sum_{d\le x}\mu(d)\left\{\frac{kx}{d}\right\}\right| .$$

From (1.27), using (2.4), we have

$$\left|\sum_{d\le x}\frac{\mu(d)}{d}\right| \le \frac{1}{kx} + \frac{1}{80}\sum_{1\le t\le k-1}\frac{1}{t} + \frac{1}{80}\,\frac{k-1}{k} + \frac{0.305}{k} + \frac{1}{160k} , \quad \text{for} \quad x > 60{,}000 .$$

Choosing k to be 20 , we have

$$\left| \sum_{d \leq x} \frac{\mu(d)}{d} \right| \leq 0.073 , \quad \text{for} \quad x > 60{,}000. \tag{2.20}$$

This suffices to complete the proof.

We have shown that, if

$$g(x) = \sum_{n \leq x} \frac{\mu(n)}{n} ,$$

then $g(x)$ assumes its minimum at $x = 13$. If we define $g_r(x)$ by

$$g_r(x) = \sum_{n \leq x} \frac{\mu(n)}{n^r} ,$$

then it is rather easy to show that, at least for integer $r \geq 2$, $g_r(x)$ assumes its minimum at $x = 5$. For

$$\sum_{n \leq x} \frac{\mu(n)}{n^r} = 1 - \frac{1}{2^r} - \frac{1}{3^r} - \frac{1}{5^r} + \frac{1}{6^r} - \frac{1}{7^r} + \frac{1}{10^r} - \frac{1}{11^r} - \frac{1}{13^r} + \frac{1}{14^r} + \cdots .$$

It is easy to see that the minimum cannot occur for $5 < x < 13$. For $r \geq 4$, we shall show that

$$\frac{1}{6^r} + \frac{1}{10^r} > \frac{1}{7^r} + \sum_{d=11}^{\infty} \frac{1}{d^r} , \tag{2.21}$$

so that the sum beyond $x = 5$ is always positive, and hence the minimum

occurs at $x = 5$.

Since

$$\sum_{d=11}^{\infty} \frac{1}{d^r} < \int_{10}^{\infty} \frac{1}{u^r}\,du = \frac{10}{r-1}\,\frac{1}{10^r} ,$$

we have

$$\begin{aligned}\frac{1}{7^r} - \frac{1}{10^r} + \sum_{d=11}^{\infty} \frac{1}{d^r} &< \frac{11-r}{r-1}\,\frac{1}{10^r} + \frac{1}{7^r} \le \frac{2\frac{1}{3}}{10^r} + \frac{1}{7^r} \\ &= \frac{2\frac{1}{3}}{7^r}\left(\frac{7}{10}\right)^r + \frac{1}{7^r} \le \frac{2\frac{1}{3}}{10^r}\left(\frac{7}{10}\right)^r + \frac{1}{7^r} \\ &< \frac{1.6}{7^r} = \frac{1.6}{6^r}\left(\frac{6}{7}\right)^r \le \frac{1.6}{6^r}\left(\frac{6}{7}\right)^4 \\ &< \frac{1}{6^r} .\end{aligned}$$

So the minimum occurs at $x = 5$ for all $r \ge 4$ (not just integer r). One can use (2.1) to obtain

$$\left| \sum_{d \le x} \frac{\mu(d)}{d^r} - \frac{1}{\zeta(r)} \right| \le \frac{1}{80}\left(2 + \frac{1}{r-1}\right)\frac{1}{x^{r-1}} , \quad \text{for } x \ge 694 \text{ and } r > 1 . \tag{2.22}$$

Using this to examine $r = 2$ and $r = 3$ we again find that the minimum occurs at $x = 5$. Indeed, it seems that there is an r_o between 1 and 2 , namely the solution of $\frac{1}{6^r} + \frac{1}{10^r} = \frac{1}{7^r} + \frac{1}{11^r} + \frac{1}{13^r}$, such

that, for $1 \leq r < r_o$ the minimum occurs at $x = 13$, for r_o there are twin minima at $x = 13$ and $x = 5$, and for $r > r_o$ the minimum occurs at $x = 5$.

CHAPTER 3

The Minimum of $\frac{\Phi(x)}{x^2}$

1. Introduction. Let $\varphi(n)$ denote Euler's function, so that

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p}) = n \sum_{d|n} \frac{\mu(d)}{d} \quad .$$

Define

$$\Phi(x) \text{ by } \Phi(x) = \sum_{d \le x} \varphi(d) \quad .$$

Then we have

$$\Phi(x) = \sum_{d \le x} d \sum_{k|d} \frac{\mu(k)}{k} = \sum_{d \le x} \mu(d) \sum_{n \le \frac{x}{d}} n$$

$$= \frac{1}{2} \sum_{d \le x} \mu(d) \, [\tfrac{x}{d}]^2 + \frac{1}{2} \tag{3.1}$$

$$= \frac{1}{2} \sum_{d \le x} \mu(d) \left(\frac{x^2}{d^2} - 2 \frac{x}{d} \{\tfrac{x}{d}\} + \{\tfrac{x}{d}\}^2 \right) + \frac{1}{2} \quad .$$

Hence,

$$|\Phi(x) - \frac{3}{\pi^2} x^2| \leq \frac{1}{2} x^2 \left| \sum_{d>x} \frac{\mu(d)}{d^2} \right| + x \left| \sum_{d\leq x} \frac{\mu(d)}{d} \{\frac{x}{d}\} \right| +$$

$$+ \frac{1}{2} \left| \sum_{d\leq x} \mu(d) \{\frac{x}{d}\}^2 \right| + \frac{1}{2} \quad . \tag{3.2}$$

Using the trivial estimates

$$\left| \sum_{d>x} \frac{\mu(d)}{d^2} \right| \leq \frac{1}{[x]} \leq \frac{1}{x} + \frac{2}{x^2} \quad ,$$

$$\left| \sum_{d\leq x} \frac{\mu(d)}{d} \{\frac{x}{d}\} \right| \leq \log x + 1 \quad ,$$

and

$$\left| \sum_{d\leq x} \mu(d) \{\frac{x}{d}\}^2 \right| \leq x \quad ,$$

we have

$$|\Phi(x) - \frac{3}{\pi^2} x^2| \leq x \log x + 2x + \frac{3}{2} \quad . \tag{3.3}$$

In particular, if $R(x) = \Phi(x) - \frac{3}{\pi^2} x^2$, then

$$R(x) = O\ (x \log x) \quad . \tag{3.4}$$

Equation (3.4) was first proved by F. Mertens [13] in 1874, and even minor improvement seems hard to come by. In a long and difficult paper A. Walfisz [32] in 1953 showed that

$$R(x) = O\left(x (\log x)^{3/4} (\log \log x)^2\right) \quad , \tag{3.5}$$

and A. Saltykov [23] in 1960 replaced $3/4$ by $2/3$ and 2 by $(1+\epsilon)$. In the opposite direction S. S. Pillai and S. D. Chowla [18] showed that

$$R(x) \neq o\left(x \log \log \log x\right) \quad , \tag{3.6}$$

and also that

$$\sum_{n \leq x} R(n) = \frac{3}{2\pi^2} x^2 + o\left(x^2\right) \quad . \tag{3.7}$$

J. J. Sylvester [28] conjectured that $R(x) > 0$, for all positive integers x, a result which holds for small x. But M. L. N. Sarma [24] showed that $R(820) < 0$, while P. Erdos and H. N. Shapiro [5] showed that $R(x)$ changes sign for infinitely many integers x, and indeed that there exists a positive constant c and infinitely many integers x such that

$$R(x) > cx \log \log \log \log x$$

and infinitely many integers x such that

$$R(x) < -cx \log \log \log \log x \quad .$$

We consider here the problem of determining the S-density of $\Phi(x)$, that is, of finding

$$S(\Phi) = \min_{\text{integer } x>0} \frac{\Phi(x)}{x^2} - \frac{3}{\pi^2} = \min_{\text{integer } x>0} \frac{R(x)}{x^2} . \tag{3.8}$$

We show that this minimum occurs at $x = 1276$ (1276 is the second integer x for which $R(x) < 0$), where $\frac{R(x)}{x^2}$ has the value $\frac{274,433}{814,088} - \frac{3}{\pi^2} = -0.2466 \ldots \times 10^{-4}$; this is done by showing that,

$$\left| \frac{\Phi(x)}{x^2} - \frac{3}{\pi^2} \right| < \left| \frac{\Phi(1276)}{(1276)^2} - \frac{3}{\pi^2} \right| , \quad \text{for } x > 150,000, \tag{3.9}$$

and then using the computer to check the remaining values of x.

2. <u>Proof of (3.9)</u> In what follows, let x be an integer. (Note: If we dropped this requirement on x, the result (3.9) would often not hold for values of x just below smaller integers. However, if instead of the step function

$$\Phi(x) = \sum_{n\leq x} \varphi(n)$$

we worked with the continuous function

$$\Phi_1(x) = \sum_{n\leq x} \varphi(n) + \{x\} \varphi([x+1]) ,$$

(3.9) would hold for all $x \neq 1276$ by fairly straight forward extensions of the work herein.)

From (3.3), we have

$$\left|\frac{\Phi(x)}{x^2} - \frac{3}{\pi^2}\right| \leq \frac{2}{x} + \frac{\log x}{x} + \frac{3}{2x^2} < 0.24 \times 10^{-4} \quad \text{for} \quad x > 650{,}000 \ . \tag{3.10}$$

We could ask the computer to check up to 650,000, but the time required increases very rapidly with x, so to save computer time we shall reduce this to 150,000 by improving the trivial result (3.3).

We have

$$\sum_{d>x} \frac{\mu(d)}{d^2} = \sum_{d>x} M(d) \left(\frac{1}{d^2} - \frac{1}{(d+1)^2}\right) - \frac{M(x)}{(x+1)^2}$$

so that, using (2.1),

$$\left| \sum_{d>x} \frac{\mu(d)}{d^2} \right| \leq \frac{1}{80} \sum_{d>x} d \left(\frac{1}{d^2} - \frac{1}{(d+1)^2}\right) + \frac{1}{80} \, \frac{1}{x}$$

$$\leq \frac{1}{40} \, \frac{1}{x} + \frac{1}{80} \, \frac{1}{x} = \frac{3}{80} \, \frac{1}{x} , \quad \text{for} \quad x \geq 1114. \tag{3.11}$$

Let

$$\sum_{d \leq x} \mu(d) \left\{\frac{x}{d}\right\}^2 = \sum_{d \leq \frac{x}{k}} \mu(d) \left\{\frac{x}{d}\right\}^2 + \sum_{\frac{x}{k} \leq d \leq x} \mu(d) \left\{\frac{x}{d}\right\}^2 \quad ,$$

where k will be an appropriately chosen integer. From (3.5) we have

$$\left|\sum_{d\le\frac{x}{k}}\mu(d)\left\{\frac{x}{d}\right\}^2\right| \le \sum_{d\le\frac{x}{k}}|\mu(d)| = Q\left(\frac{x}{k}\right) < 0.615\,\frac{x}{k}\,, \quad \text{for} \quad x \ge 475\,k \;. \tag{3.12}$$

Also,

$$\sum_{\frac{x}{k}< d\le x}\mu(d)\left\{\frac{x}{d}\right\}^2 = \sum_{\frac{x}{k}< d\le x-1}\mu(d)\left(\left\{\frac{x}{d}\right\}-\left\{\frac{x}{d+1}\right\}\right)\left(\left\{\frac{x}{d}\right\}+\left\{\frac{x}{d+1}\right\}\right)$$

$$- M\left(\frac{x}{k}\right)\left\{\frac{x}{\left[\frac{x}{k}\right]+1}\right\}^2 \;.$$

Now, for $\frac{x}{k} < d$, we have $\left[\frac{x}{d}\right] \le k-1$, so that $\left[\frac{x}{d}\right] \neq \left[\frac{x}{d+1}\right]$, for at most k values of d, or, equivalently, $\left\{\frac{x}{d}\right\} - \left\{\frac{x}{d+1}\right\} = \frac{x}{d(d+1)}$ with at most k exceptions. Also, $\left|\left\{\frac{x}{d}\right\} + \left\{\frac{x}{d+1}\right\}\right| < 2$, and for $\frac{x}{d} < d \le x-1$ we can use the result that $|M(d)| < \frac{1}{2}\sqrt{d}$ for $\frac{x}{d} \ge 200$ and $x \le 10^8$, i.e. for $200\,k \le x \le 10^8$. As our k will be 25, this gives

$$|M(d)| < \frac{1}{2}\sqrt{d}\,, \quad \text{for} \quad 5000 \le x \le 10^8 \;.$$

Thus, we have

$$\left| \sum_{\frac{x}{k} < d \leq x} \mu(d) \{\tfrac{x}{d}\}^2 \right| \leq 2x \sum_{\frac{x}{k} < d \leq x-1} \frac{|M(d)|}{d(d+1)} + k \cdot \frac{1}{2}\sqrt{x} + |M(\tfrac{x}{k})|$$

$$< x \sum_{\frac{x}{k} < d \leq x-1} \frac{1}{\sqrt{d}\ (d+1)} + \frac{1}{2} k \sqrt{x} + \frac{1}{2\sqrt{k}} \sqrt{x}$$

$$\leq 2x \frac{\sqrt{k}-1}{\sqrt{x}} + \frac{1}{2} k \sqrt{x} + \frac{1}{2\sqrt{k}} \sqrt{x}$$

$$= \sqrt{x} \left(\frac{k}{2} + 2\sqrt{k} - 2 + \frac{1}{2\sqrt{k}}\right), \quad \text{for} \quad 5000 \leq x \leq 10^8 . \qquad (3.13)$$

Therefore,

$$\left| \sum_{d \leq x} \mu(d) \{\tfrac{x}{d}\}^2 \right| < 0.615 \frac{x}{k} + \sqrt{x} \left(\frac{k}{2} + 2\sqrt{k} - 2 + \frac{1}{2\sqrt{k}}\right)$$

$$\text{for} \quad 5000 \leq x \leq 10^8 \quad \text{and} \quad x \geq 475\, k \quad .$$

When $k = 25$, the right hand side becomes $0.0246\, x + 20.6 \sqrt{x}$, and

$$\left| \sum_{d \leq x} \mu(d) \{\tfrac{x}{d}\}^2 \right| < 0.078\, x , \quad \text{for} \quad 150{,}000 \leq x \leq 10^8 . \qquad (3.14)$$

We now consider the sum

$$\sum_{d \leq x} \frac{\mu(d)}{d} \{\tfrac{x}{d}\} \quad .$$

Writing

$$A_1(x) = \sum_{\substack{d<x \\ \mu(d)=1}} \frac{1}{d}\ , \quad B_1(x) = \sum_{\substack{d<x \\ \mu(d)=-1}} \frac{1}{d}\ , \quad \text{and} \quad C_1(x) = \max\,(A_1(x), B_1(x))\ ,$$

we see that

$$\left| \sum_{d\le x} \frac{\mu(d)}{d} \left\{\frac{x}{d}\right\} \right| \le C_1(x) \quad . \tag{3.15}$$

Now, $C_1(x) = \frac{1}{2}\,(A_1(x) + B_1(x) + |A_1(x) - B_1(x)|)$, so that

$$C_1(x) = \frac{1}{2} \sum_{d\le x} \frac{|\mu(d)|}{d} + \frac{1}{2} \left| \sum_{d\le x} \frac{\mu(d)}{d} \right| \quad . \tag{3.16}$$

From (2.15) we had

$$\left| \sum_{d\le x} \frac{\mu(d)}{d} \right| < 0.036 \quad \text{for} \quad 900 \le x \le 10^8 \quad .$$

For $x \ge 900$ we have

$$\sum_{d\le x} \frac{|\mu(d)|}{d} = \sum_{d\le 900} \frac{|\mu(d)|}{d} + \sum_{900<d\le x-1} \frac{Q(d)}{d(d+1)} + \frac{Q(x)}{x} - \frac{Q(900)}{901} \quad .$$

Using (2.5), and observing that

$$\sum_{d\le 900} \frac{|\mu(d)|}{d} = 5.178 \ldots \quad \text{and} \quad Q(900) = 547 \quad ,$$

we have

$$\sum_{d \le x} \frac{|\mu(d)|}{d} \le 0.615 \log x + 1.012 \ , \quad \text{for} \quad x \ge 900 \ . \tag{3.17}$$

Combining (2.15) and (3.17) in (3.16), we obtain

$$C_1(x) \le 0.3075 \log x + 0.506 \ , \quad \text{for} \quad 900 \le x \le 10^8 \ , \tag{3.18}$$

and

$$\left| \sum_{d \le x} \frac{\mu(d)}{d} \left\{\frac{x}{d}\right\} \right| \le 0.3075 \log x + 0.506 \ , \quad \text{for} \quad 900 \le x \le 10^8 \ . \tag{3.19}$$

Unfortunately, (3.19) is not quite strong enough to give us our required result, and we must proceed as follows: for $k = 25$, (3.18) yields

$$\begin{aligned} \left| \sum_{d \le \frac{x}{k}} \frac{\mu(d)}{d} \left\{\frac{x}{d}\right\} \right| &\le 0.3075 \log \frac{x}{25} + 0.506 \\ &< 0.3075 \log x - 0.482, \quad \text{for} \quad 22500 \le x \le 10^8 \ . \end{aligned} \tag{3.20}$$

On the other hand,

$$\sum_{\frac{x}{k} < d \le x} \frac{\mu(d)}{d} \left\{\frac{x}{d}\right\} = \sum_{\frac{x}{k} < d \le x-1} M(d)\left(\left\{\frac{x}{d}\right\}/d - \left\{\frac{x}{d+1}\right\}/(d+1)\right) - M\left(\frac{x}{k}\right) \frac{\left\{\frac{x}{[\frac{x}{k}]+1}\right\}}{[\frac{x}{k}]+1} \ .$$

Now,

$$\{\tfrac{x}{d}\}/d - \{\tfrac{x}{d+1}\}/(d+1) = (\{\tfrac{x}{d}\} - \{\tfrac{x}{d+1}\})/(d+1) + \{\tfrac{x}{d}\}/d(d+1) \quad .$$

As above, we note that $\{\frac{x}{d}\} - \{\frac{x}{d+1}\} = \frac{x}{d(d+1)}$ except for at most k values of d. For these exceptional values

$$|M(d)(\{\tfrac{x}{d}\} - \{\tfrac{x}{d+1}\})|/(d+1) < \frac{|M(d)|}{d+1} < \frac{1}{2}\frac{1}{\sqrt{x/k}} = \frac{\sqrt{k}}{2\sqrt{x}} \quad ,$$

where we use the fact that $M(d) < \frac{1}{2}\sqrt{d}$ for $\frac{x}{25} > 200$ and $x \le 10^8$, i.e. for $5000 \le x \le 10^8$. Thus,

$$\Big|\sum_{\frac{x}{k} < d \le x} \frac{\mu(d)}{d}\{\tfrac{x}{d}\}\Big| \le x \sum_{\frac{x}{k} < d \le x-1} \frac{|M(d)|}{d(d+1)^2} + \frac{k^{3/2}}{2\sqrt{x}}$$

$$+ \sum_{\frac{x}{k} < d \le x-1} \frac{|M(d)|}{d(d+1)} + |M(\tfrac{x}{k})|/\tfrac{x}{k}$$

$$\le \frac{1}{2}x \sum_{\frac{x}{k} < d \le x-1} \frac{1}{\sqrt{d}(d+1)^2} + \frac{1}{2}\sum_{\frac{x}{k} < d \le x-1} \frac{1}{\sqrt{d}(d+1)}$$

$$+ \frac{k^{3/2}}{2\sqrt{x}} + \frac{\sqrt{k}}{2\sqrt{x}}$$

$$\le \frac{(k^{3/2} - 1)}{3\sqrt{x}} + \frac{(\sqrt{k} - 1)}{\sqrt{x}} + \frac{k^{3/2}}{2\sqrt{x}} + \frac{\sqrt{k}}{2\sqrt{x}} \quad .$$

Substituting $k = 25$, we get

$$\left| \sum_{\frac{x}{k} < d \le x} \frac{\mu(d)}{d} \left\{\frac{x}{d}\right\} \right| \le \frac{332}{3\sqrt{x}} , \quad \text{for} \quad 5000 < x \le 10^8 ,$$

$$< 0.287 , \quad \text{for} \quad 150{,}000 \le x \le 10^8 , \qquad (3.21)$$

so that, with (3.20), we have

$$\left| \sum_{d \le x} \frac{\mu(d)}{d} \left\{\frac{x}{d}\right\} \right| \le 0.3075 \log x - 0.195 . \qquad (3.22)$$

Hence, using (3.22), (3.14), and (3.11) in (3.2) we have

$$\left| \frac{\Phi(x)}{x^2} - \frac{3}{\pi^2} \right| \le 0.3075 \frac{\log x}{x} - \frac{0.079}{x} + \frac{1}{2x^2}$$

$$< 0.24 \times 10^{-4} , \quad \text{for} \quad 150{,}000 \le x \le 10^8 . \quad (3.23)$$

So now we have only to check for $x < 150{,}000$.

3. The computer work. Theorems 3.1 and 3.2 are essentially due to R. S. Lehman [11], who gave the results for $h(n) = \lambda(n) = (-1)^r$, where $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$.

Theorem 3.1. Let $H(x) = \sum_{n \le x} h(n)$, $f(x) = \sum_{n \le x} H(\frac{x}{n})$. Then

$$H(x) = \sum_{m \le x/w} \mu(m) \left\{ f\left(\frac{x}{m}\right) - \sum_{k < v} h(k)\left(\left[\frac{x}{km}\right] - \left[\frac{x}{vm}\right]\right)\right\} - \sum_{x/w < \ell \le x/v} H\left(\frac{x}{\ell}\right) \sum_{\substack{m \mid \ell \\ m \le x/w}} \mu(m) ,$$

(3.24)

where $1 < v < w < x$.

<u>Proof</u>. $$f\left(\frac{x}{m}\right) = \sum_{n \le x/mw} H\left(\frac{x}{mn}\right) + \sum_{x/mw < n \le x/mv} H\left(\frac{x}{mn}\right) + \sum_{x/mv < n \le x/m} H\left(\frac{x}{mn}\right) .$$

Multiply each by $\mu(m)$, and sum for $m \le \frac{x}{w}$.

1. $$\sum_{m \le x/w} \mu(m) \sum_{n \le x/mw} H\left(\frac{x}{mn}\right) = \sum_{\ell \le x/w} H\left(\frac{x}{\ell}\right) \sum_{m|\ell} \mu(m) = H(x) .$$

2. $$\sum_{m \le x/w} \mu(m) \sum_{x/mw < n \le x/mv} H\left(\frac{x}{mn}\right) = \sum_{x/w < \ell \le x/v} H\left(\frac{x}{\ell}\right) \sum_{\substack{m|\ell \\ m \le x/w}} \mu(m) .$$

3. $$\sum_{m \le x/w} \mu(m) \sum_{x/mv < n \le x/m} H\left(\frac{x}{mn}\right) = \sum_{m \le x/w} \mu(m) \sum_{x/mv < n \le x/m} \sum_{k \le x/mn} h(k)$$

$$= \sum_{m \le x/w} \mu(m) \sum_{k < v} h(k) \sum_{x/mv < n \le x/km} 1$$

$$= \sum_{m \le x/w} \mu(m) \sum_{k < v} h(k) \left(\left[\frac{x}{km}\right] - \left[\frac{x}{vm}\right]\right) .$$

Rearrangement of terms then yields the theorem. We note that, by choosing $v \approx x^{1/3}$ and $w \approx x^{2/3}$, if we have tables of $\mu(m)$ and $h(m)$ for $m \le v$, $H(m)$ for $m \le w$, and $\xi(\ell) = \sum_{\substack{m|\ell \\ m \le x/w}} \mu(m)$ for

$\ell \leq \frac{x}{v}$, then the number of operations is proportional to $x^{2/3}$. The last-mentioned tables can readily be computed as the program progresses by adding $\mu([\frac{x}{w}])$ to each $[\frac{x}{w}]$-th value of ξ whenever $[\frac{x}{w}]$ increases by 1 .

The following slightly more complicated formula is in practice more efficient, cutting the time by about one half.

Theorem 3.2. Let

$$f(x) = \sum_{n \leq x} H(\frac{x}{n}) \quad \text{and} \quad H(x) = \sum_{n \leq x} h(n) \quad .$$

Let K' , L', and M' range over positive odd integers, and I over positive integers. Then

$$H(N) = \sum_{M' \leq N/w} \mu(M') \left\{ f(\frac{N}{M'}) - f(\frac{N}{2M'}) - \left(\sum_{I<v} h(I) \left(\frac{N+IM'}{2IM'}\right) - \frac{N+vM'}{2vM'} H(v-1) \right) \right\}$$

$$- \sum_{N/w+1 \leq K' \leq N/v} H(\frac{N}{K'}) \sum_{\substack{L' | K' \\ L' \leq N/w}} \mu(L') \ , \quad \text{where} \quad 1 < v < w < N \ . \qquad (3.25)$$

Proof. From $f(x) = \sum_{n \leq x} H(\frac{x}{n})$ we get $f(x) - f(\frac{x}{2}) = \sum_{M' \leq x} H(\frac{x}{M'})$.

The proof now follows as in Theorem 3.1, if we note that, if $k = 2^t \ell$, where $(\ell, 2) = 1$, then

$$\sum_{\substack{M' \mid k \\ (M' \text{ odd})}} \mu(M') = \sum_{m \mid \frac{k}{2^t}} \mu(m) = \begin{cases} 1, & \text{if } \ell = 1\,, \\ 0, & \text{if } \ell > 1\,, \end{cases}$$

so that, if k is odd,

$$\sum_{\substack{M' \mid k \\ (M' \text{ odd})}} \mu(M') = [\tfrac{1}{k}] \quad ;$$

and also that the number of odd integers $\leq x$ is $[\frac{x+1}{2}]$.

Corollary.

$$\Phi(N) = \sum_{M' \leq N/w} \mu(M') \left\{ \frac{[\frac{N}{M'}]\,[\frac{N}{M'}+1]}{2} - \frac{[\frac{N}{2M'}]\,[\frac{N}{2M'}+1]}{2} - \left(\sum_{I \leq N-1} \varphi(I)\left(\frac{N+IM'}{2IM'}\right) \right.\right.$$

$$\left.\left. - \frac{N+vM'}{2vM'}\,\Phi(v-1)\right)\right\} - \sum_{N/w+1 \leq K' \leq N/v} \Phi(\tfrac{N}{K'}) \sum_{\substack{L' \mid K' \\ L' \leq N/w}} \mu(L') \,. \qquad (3.26)$$

For

$$\sum_{n \leq x} \Phi(\tfrac{x}{n}) = \sum_{n \leq x}\ \sum_{m \leq x/n} \varphi(m) = \sum_{n \leq x} \varphi(n)\ [\tfrac{x}{n}]$$

$$= \sum_{n \leq x} [\tfrac{x}{n}]\ n \sum_{d \mid n} \frac{\mu(d)}{d} = \sum_{d \leq x} d \sum_{k \leq x/d} \mu(k)\ [\tfrac{x}{kd}]$$

$$= \sum_{d \leq x} d = \frac{[x][x+1]}{2} \quad .$$

A preliminary program was used to obtain $\mu(n)$ and $\varphi(n)$ for $n = 1$ to 60 and $\Phi(n)$ for $n = 1$ to 3600 on cards as data to be read into the machine for the main program based on (3.26)

We note that, since we are just interested in determining the minimum value of $\frac{\Phi(x)}{x^2}$, we can reduce the checking, and hence the computer time, by observing

$$\frac{\Phi(2n)}{(2n)^2} < \frac{\Phi(2n-1)}{(2n-1)^2} \quad , \tag{3.27}$$

$$\frac{\Phi(6n-2)}{(6n-2)^2} < \frac{\Phi(6n-4)}{(6n-4)^2} \quad , \tag{3.28}$$

and

$$\frac{\Phi(30n-24)}{(30n-24)^2} < \frac{\Phi(30n-26)}{(30n-26)^2} \quad . \tag{3.29}$$

We prove (3.27) ; the others are proven similarly. Now,

$$\frac{\Phi(2n)}{(2n)^2} < \frac{\Phi(2n-1)}{(2n-1)^2} \quad \text{if and only if} \quad \frac{\varphi(2n)}{(2n)^2} < \Phi(2n-1)\left(\frac{1}{(2n-1)^2} - \frac{1}{(2n)^2}\right),$$

i.e. , if and only if, $\varphi(2n) < \Phi(2n-1)\ \frac{4n-1}{(2n-1)^2}$. Hence it suffices to prove

$$\varphi(2n) < \frac{2\Phi(2n-1)}{2n-1} \quad .$$

But $\varphi(2n) \leq n-1$, while by (3.3) ,

$$\frac{2\Phi(2n-1)}{2n-1} > \frac{6}{\pi^2}(2n-1) - 2\log(2n-1) - 2 - \frac{3}{2(2n-1)}$$

$$= \frac{12}{\pi^2} n - 2\log(2n-1) - (2 + 6/\pi^2) - \frac{3}{2(2n-1)}$$

$$> n \quad \text{for} \quad n \geq 61 ;$$

and (3.27) is readily verified for $n = 1, 2, \ldots, 60$.

Hence, we need only examine nine residue classes modulo 30 , namely 0, 6, 10, 12, 16, 18, 22, 24, 28 .

We were able to determine all values of $n \leq 150{,}000$ for which $R(n)$ is negative. These 286 values are given in Appendix II.

APPENDIX I

Fortran IV programs for examining $\frac{1}{x^2} \sum_{n \leq x} \varphi(n) - \frac{3}{\pi^2}$.

1. $\mu(n)$ on cards for $1 \leq n \leq 60$. Uses $\sum_{n \leq x} M(\frac{x}{n}) = 1$.

```
      DIMENSION MU(60), MBIG(60)
      MU(1) = 1
      MBIG(1) = 1
      DO 10 I = 2,60
      NS = 0
      L = I - 1
      DO 20 J = 2, I
      K = I/J
      NS = NS + MBIG(K)
20    CONTINUE
      MBIG(I) = 1 - NS
      MU(I) = MBIG(I) - MBIG(L)
10    CONTINUE
      WRITE (7,30) (MU(N), N = 1,60)
30    FORMAT (40I2)
      STOP
      END
```

2. $\Phi(n)$ and $\varphi(n)$ on cards. $\Phi(n)$ for $1 \leq n \leq 3600$, $\varphi(n)$ for $1 \leq n \leq 60$.

Uses $\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$.

```
      DIMENSION  NPR(505), LPHI(3600), NPHI(3600)
      NPR(1) = 2
      NPR(2) = 3
      DO 20 I = 2,506
      NPR(I+1) = NPR(I)+2
      X = NPR(I+1)
      NSR = SQRT(X)
11    DO 20 J = 2,NSR
      IF  (NPR(I+1) - (NPR(I+1)/NPR(J)) * NPR(J)) 20,10,20
10    NPR(I+1) = NPR(I+1)+2
      X = NPR(I+1)
      NSR = SQRT(X)
      GO TO 11
20    CONTINUE
      NPHI(1) = 1
      LPHI(1) = 1
      N = 2
43    LPHI(N) = N
      DO 35 I = 1,N
      IF  (NPR(I)-N) 38,39,40
```

```
38    IF  (N-(N/NPR(I)) * NPR(I)) 35,34,35
34    LPHI(N) = LPHI(N) * (NPR(I)-1)/NPR(I)
35    CONTINUE
39    LPHI(N) = N-1
40    NPHI(N) = NPHI(N-1)+LPHI(N)
78    N = N+1
      IF  (N-3600) 81,82,82
81    GO TO 43
82    WRITE (7,41) (LPHI(J),J = 1,60)
41    FORMAT (10I8)
      WRITE (7,41) (NPHI(J),J = 1,3600)
      STOP
      END
```

3. $\Phi(x)$ using equation (3.26)

```
      DIMENSION  LPHI(60), NPHI(3600), MU(60), NS3(3600)
      READ (5,42) (MU(J),J = 1,60)
42    FORMAT (40I2)
      READ (5,41) (LPHI(J),J = 1,60)
41    FORMAT (10I8)
      READ (5,41) (NPHI(J),J = 1,3600)
      READ (5,49) N
49    FORMAT (1X,I6)
      N1 = N/3600
```

```
      DO 89 K = 1,3600
89    NS3(K) = 0
      DO 90 K = 1,N1,2
      Do 91 L = K,3600,K
91    NS3(L) = NS3(L)+MU(K)
90    CONTINUE
      IND1 = N-N1 * 3600
      INDEX = 0
43    N1 = N/3600
      N2 = N/60
      N3 = N1+1
      N3 = 1+(N3/2) * 2
      NS1 = 0
      NS4 = 0
      DO 20 M = 1,N1,2
      NS2 = 0
      DO 10 I = 1,59
      NS2 = NS2+LPHI(I) * (N+I*M)/(2*I*M))
10    CONTINUE
      NS2 = NS2-NPHI(59) * ((N+60*M)/(120*M))
      NS1 = NSI+MU(M) * (((N/M)*((N/M)+1)-N/(2*M))*((N/(2*M))+1)/2-NS2)
20    CONTINUE
      DO 30 K = N3,N2,2
      J = N/K
      NS4=NS4+NPHI(J) * NS3(K)
```

```
30    CONTINUE
      NP = NSI-NS4
      V = 2 * NP-1
      Q = N * N
      DIFF = (V/Q)-0.607927102
77    WRITE (6,80) N,NP,DIFF
80    FORMAT (1X,I6,I12,E20.10)
78    INDEX = INDEX+1
      IF (INDEX-100) 59,60,59
60    WRITE (6,85) N
85    FORMAT (1X,I6)
      INDEX = 0
59    N = N+1
      IND1 = IND1+1
      IF (IND1-3600) 96,97,96
97    LMN = N1+1
      IF (LMN - (LMN/2)*2) 93,94,93
93    DO 98 M = LMN,3600,LMN
98    NS3(M) = NS3(M)+MU(LMN)
94    IND1 = 0
96    CONTINUE
      IF (N-150000) 81,82,82
81    GO TO 43
82    STOP
      END
```

APPENDIX II

Values of n for which $R(n) < 0$.

820	10,836	21,376	32,046	42,966	54,132	65,200
1,276	13,146	21,726	32,166	43,386	55,476	65,950
1,422	13,300	22,270	32,230	43,590	55,540	67,210
1,926	15,640	22,480	32,320	43,776	56,190	67,510
2,080	15,666	22,716	32,530	43,780	57,036	67,926
2,640	16,056	23,530	35,796	44,256	57,786	68,766
3,160	16,060	25,026	36,366	45,696	58,026	69,126
3,186	16,446	25,236	36,456	45,816	58,906	69,336
3,250	17,020	25,300	36,466	46,326	59,046	69,756
4,446	17,466	25,930	36,520	48,280	59,556	70,092
4,720	17,550	26,202	36,576	48,336	60,516	70,176
4,930	17,766	26,680	37,116	48,400	60,606	70,500
5,370	18,040	27,406	37,480	48,610	61,020	70,840
6,006	18,910	27,940	38,556	49,050	61,986	70,876
6,546	19,176	28,260	38,676	51,040	62,496	71,316
7,386	19,230	28,276	39,096	51,130	62,700	71,646
7,450	19,416	28,596	41,140	51,340	62,910	71,830
7,476	20,736	29,736	41,406	52,690	63,196	72,066
9,066	21,000	30,486	41,616	52,900	63,310	73,326
9,276	21,246	31,032	41,706	52,956	64,296	73,360
10,626	21,310	31,452	42,960	53,586	64,506	73,516

74,140	87,066	104,470	114,466	125,260	138,726
74,350	88,656	104,676	114,570	125,616	139,296
74,376	88,716	105,096	114,640	126,010	140,400
76,896	89,586	105,876	114,940	127,270	140,526
77,236	90,000	105,966	115,116	127,590	141,156
77,316	90,006	106,086	115,300	127,636	141,430
77,470	90,630	106,120	117,096	129,186	142,836
78,156	91,120	106,176	117,250	129,396	143,320
78,456	91,806	106,386	117,306	131,496	143,706
80,136	92,106	106,746	118,660	131,706	144,160
80,446	92,170	107,866	119,080	131,770	145,176
80,466	92,226	108,670	119,290	132,280	145,236
80,620	92,346	108,790	119,770	134,046	145,390
80,886	92,856	109,840	120,156	134,200	145,986
81,996	93,250	111,100	122,200	134,850	147,280
82,446	94,810	111,400	122,556	134,856	148,060
83,380	94,836	111,420	122,760	135,850	148,296
83,590	95,082	111,816	122,916	136,060	149,710
85,086	96,426	112,596	123,306	136,216	149,856
85,296	96,580	112,750	123,370	136,326	150,606
85,360	98,946	113,016	123,490	136,830	150,690
85,596	101,830	113,050	123,970	137,436	
85,816	101,916	113,646	124,776	137,866	
86,346	104,380	114,192	125,196	138,546	

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